

ORTHOGONAL AND OBLIQUE PROJECTORS AND THE CHARACTERISTICS OF PAIRS OF VECTOR SPACES

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1. *Introduction.* The statistical operation of multiple linear regression by least squares is equivalent to the orthogonal projection of vectors of observations on a space spanned by vectors of observations; and a partial regression can be similarly represented as an oblique projection. This connexion between a statistical and a formal algebraical operation gives the main source of interest for this investigation. Its object is to develop algebraical theory which supplies terms necessary for a unified algebraical and geometrical formulation of concepts in multivariate analysis. An exposition of the statistical theory is to be made in a separate paper.

The expression of a multiple regression as an orthogonal projection on a linear space was first found to lead to an advantageous exposition of a statistical analysis in which several groups of intercorrelated factors were distinguished. Then it appeared that different topics in multivariate analysis, for example, the canonical correlation theory of Hotelling (4), could be translated into the algebraical theory of projectors, and also that, in this algebraical formulation, it was possible to introduce concepts fitting certain intuitive ideas which, though needed in the subject, were without any clear definition. Thus there is obtained a definition for a coefficient of correlation between factors each of which has a multiplicity of components, which reduces to the correlation coefficient of Pearson in the case of simple factors. The genesis of the other concepts is to be found in consideration of psychological and other such kinds of measurements and tests. With the idea that a test is to give a character to the individual, there is the idea that the character stands outside the test, or that several tests with distinct definition may be directed to the same feature of character, for example, intelligence, and, by this intention, therefore have a relation of equivalence. There should be, accordingly, a theory of the equivalence of tests, and, as a further example, a theory of the inclusion of tests, by which one test can be decided as a proper refinement of another, or as comprehending essentially more features of character than another. Again, there is to be decided what features tests may have in common, if any. Generally, a number of tests together may be regarded as forming a configuration, from which is derived different measures for different relations the tests may have to one another. In translation to the algebraical scheme, there are various coefficients defined which belong to the configuration which vector subspaces of a given space form together; and an account of such coefficients is given in this paper, with motive towards their statistical application.

2. *Vector spaces.* Let $\mathfrak{L}$ be the space of vectors of order n formed of the real numbers. Then, with the standard laws of composition for vectors, $\mathfrak{L}$ is a linear space of dimen-

sion n over the real numbers. Any set of k vectors in $\mathfrak{L}$ is taken to compose a matrix $\mathbf{K}$ of order $n \times k$, which may be called a vector array in $\mathfrak{L}$, with these vectors as its elements; and the span of these vectors is the subspace $\mathfrak{f}$ of $\mathfrak{L}$ of vectors $\mathbf{X} = \mathbf{K}\mathbf{x}$, having the k vectors as generators. The maximal sets of independent vectors of the space $\mathfrak{f}$ all have the same number of members, which defines the dimension of the space; it is equal to the maximum number of independent vectors among the generators, and to the rank of the generating matrix $\mathbf{K}$.

THEOREM 1.1. *The rank of $\mathbf{K}$ is equal to the rank of $\mathbf{K}'\mathbf{K}$.*

For if $\mathbf{K}\mathbf{x} = \mathbf{0}$, then $\mathbf{K}'\mathbf{K}\mathbf{x} = \mathbf{0}$; and if $\mathbf{K}'\mathbf{K}\mathbf{x} = \mathbf{0}$, then $\mathbf{x}'\mathbf{K}'\mathbf{K}\mathbf{x} = \mathbf{0}$, which implies $\mathbf{K}\mathbf{x} = \mathbf{0}$; therefore $\mathbf{K}$, $\mathbf{K}'\mathbf{K}$ have the same column nullity, and, since they have the same number of columns, the same rank.

COROLLARY. *The vectors of $\mathbf{K}$ are independent if and only if $|\mathbf{K}'\mathbf{K}| \neq 0$.*

Let $\mathfrak{E}$ be a subspace of $\mathfrak{L}$ of dimension p , and $\mathbf{E}$ the array of a maximal set of independent vectors of $\mathfrak{E}$, any such set having p members. Then $\mathbf{E}$ defines a *base* for $\mathfrak{E}$, such that the vectors of $\mathfrak{E}$ are the vectors uniquely given in the form $\mathbf{X} = \mathbf{E}\mathbf{x}$; thus $\mathbf{E}\mathbf{x} = \mathbf{E}\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$, and $|\mathbf{E}'\mathbf{E}| \neq 0$, so that the matrix $(\mathbf{E}'\mathbf{E})^{-1}$ exists. There is thus a coordination between the vectors $\mathbf{X}$ of the space $\mathfrak{E}$ of dimension p , and the vectors $\mathbf{x}$ of order p , the coordination being defined relative to the base $\mathbf{E}$. The following results are immediate.

THEOREM 1.2. *A necessary and sufficient condition for a vector $\mathbf{X}$ to belong to a space $\mathfrak{E}$ with base $\mathbf{E}$ is that $\mathbf{I}_{\mathbf{E}}\mathbf{X} = \mathbf{X}$, where $\mathbf{I}_{\mathbf{E}} = \mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}\mathbf{E}'$, and then $\mathbf{X} = \mathbf{E}\mathbf{x}$, where $\mathbf{x} = (\mathbf{E}'\mathbf{E})^{-1}\mathbf{E}'\mathbf{X}$; and a necessary and sufficient condition that $\mathbf{X}$ belongs to the orthogonal complement of $\mathfrak{E}$ is $\mathbf{I}_{\mathbf{E}}\mathbf{X} = \mathbf{0}$, that is, $\mathbf{J}_{\mathbf{E}}\mathbf{X} = \mathbf{X}$, where $\mathbf{J}_{\mathbf{E}} = \mathbf{I} - \mathbf{I}_{\mathbf{E}}$. Moreover, if p is the dimension of $\mathfrak{E}$, then $p = \text{rank } \mathbf{I}_{\mathbf{E}} = \text{trace } \mathbf{I}_{\mathbf{E}}$.*

An orthonormal base $\mathbf{U}$ for $\mathfrak{E}$ is composed of a set of p orthogonal unit vectors; so $\mathbf{U}'\mathbf{U} = \mathbf{I}$ if $\mathbf{U}$ is an orthonormal base, and $\mathbf{I}_{\mathbf{U}} = \mathbf{U}\mathbf{U}'$.

Let $\mathfrak{E}$, $\mathfrak{F}$ be any spaces in $\mathfrak{L}$ of dimension p , q and with bases $\mathbf{E}$, $\mathbf{F}$. The spaces, and, correspondingly, also the bases, are said to be *separate* if the intersection $\mathfrak{E} \cap \mathfrak{F}$ is the null space, which is the case if and only if the join $(\mathbf{E}\mathbf{F})$ of the bases, the span of which is the join of the spaces, denoted by $\mathfrak{E} \oplus \mathfrak{F}$, is also a base; otherwise they are said to be *incident*. They are called *complementary* if they are separate and their union is the complete space $\mathfrak{L}$; in this case $p + q = n$, and the join of the bases is a regular square matrix. The spaces are *orthogonal* if every vector in one is orthogonal to every vector in the other, the condition for which is $\mathbf{I}_{\mathbf{E}}\mathbf{I}_{\mathbf{F}} = \mathbf{0}$, or equivalently, $\mathbf{I}_{\mathbf{F}}\mathbf{I}_{\mathbf{E}} = \mathbf{0}$; otherwise they are said to be *inclined*. If $\mathfrak{E} \subset \mathfrak{F}$, in which case $\mathfrak{E}$ is said to be *included* in $\mathfrak{F}$, then there exists a matrix α of order $q \times p$ such that $\mathbf{E} = \mathbf{F}\alpha$, and conversely; and a necessary and sufficient condition for this is $\mathbf{I}_{\mathbf{F}}\mathbf{I}_{\mathbf{E}} = \mathbf{I}_{\mathbf{E}}$, which implies

$$\mathbf{E} = \mathbf{F}\alpha \quad \text{with} \quad \alpha = (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{E}.$$

Accordingly the spaces are identical, $\mathfrak{E} = \mathfrak{F}$, just if $\mathbf{I}_{\mathbf{E}} = \mathbf{I}_{\mathbf{F}}$; also they are orthogonal complements just if $\mathbf{I}_{\mathbf{E}} + \mathbf{I}_{\mathbf{F}} = \mathbf{I}$. They are said to be *orthogonally incident* if the orthogonal complements of their intersection in each of them are orthogonal; and this is just $\mathbf{I}_{\mathbf{E}}\mathbf{I}_{\mathbf{F}} = \mathbf{I}_{\mathbf{F}}\mathbf{I}_{\mathbf{E}}$. The space $\mathfrak{E}$ is said to be *completely inclined* to the space $\mathfrak{F}$ if no

subspace of $\mathfrak{E}$ is orthogonal to $\mathfrak{F}$, $\mathfrak{E} \ominus \mathfrak{F} = \mathfrak{D}$; and spaces are said to be *totally inclined* if they are completely inclined to each other.

Two bases U, V are said to form a *canonical pair* if each is orthonormal, and every vector of one is inclined to at most one vector of the other and orthogonal to the rest; in this case

$$U'U = 1, \quad V'V = 1, \quad U'V = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix},$$

where ρ is a regular diagonal matrix, with elements between -1 and 1 . It will appear that for every pair of spaces there exists such a canonical pair of bases.

3. *Determinants.* The following results on determinants will be required.

LEMMA 3.1. *If A, B are any matrices of order $m \times n, n \times m$, then*

$$|I - AB| = |I - BA|.$$

Proof. Take determinants in the identity

$$\begin{pmatrix} I - AB & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I - BA \end{pmatrix}.$$

COROLLARY 3.11. $\lambda^n | \lambda I - AB | = \lambda^m | \lambda I - BA |$.

COROLLARY 3.12. *The matrices AB, BA have the same non-zero characteristic values, with the same multiplicities; and zero has the same multiplicity as a characteristic value of each if and only if $m = n$.*

LEMMA 3.2. *If A is a regular square matrix, then*

$$\begin{vmatrix} A & V \\ U & B \end{vmatrix} = |A| |B - UA^{-1}V|.$$

Proof. Take determinants in the identity

$$\begin{pmatrix} I & 0 \\ -UA^{-1} & I \end{pmatrix} \begin{pmatrix} A & V \\ U & B \end{pmatrix} \begin{pmatrix} I & -A^{-1}V \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B - UA^{-1}V \end{pmatrix}.$$

COROLLARY. *If A, B are regular square matrices, then*

$$\begin{vmatrix} A & V \\ U & B \end{vmatrix} = |A| |B| |I - UA^{-1}VB^{-1}|.$$

THEOREM 3.3. *If G is the join of bases E, F , then*

$$|G'G| / |E'E| |F'F| = |E'J_F E| / |E'E| = |F'J_E F| / |F'F| = |I - I_E I_F|.$$

For

$$|G'G| = \begin{vmatrix} E'E & E'F \\ F'E & F'F \end{vmatrix} = |E'E| |F'F - F'E(E'E)^{-1}E'F| = |E'E| |F'J_E F|,$$

by Lemma 3.2. Then, by Corollary 3.21 and by Lemma 3.1,

$$\begin{aligned} |G'G| &= |E'E| |F'F| |I - F'E(E'E)^{-1}E'F(F'F)^{-1}| \\ &= |E'E| |F'F| |I - I_E I_F|. \end{aligned}$$

COROLLARY. The three conditions $|\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E}| \neq 0$, $|\mathbf{F}'\mathbf{J}_\mathbf{E}\mathbf{F}| \neq 0$, $|\mathbf{I} - \mathbf{I}_\mathbf{E}\mathbf{I}_\mathbf{F}| \neq 0$ are equivalent, and are necessary and sufficient for the bases $\mathbf{E}$, $\mathbf{F}$ to be separate.

For $\mathbf{G} = (\mathbf{E}\mathbf{F})$ is a base if and only if $\mathbf{E}$, $\mathbf{F}$ are separate bases.

THEOREM 3.4.

$$(-\rho)^{2n} \begin{vmatrix} -\rho\mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho\mathbf{F}'\mathbf{F} \end{vmatrix} = (-\rho)^{p+q} |\mathbf{E}'\mathbf{E}| |\mathbf{F}'\mathbf{F}| |\rho^2\mathbf{I} - \mathbf{I}_\mathbf{E}\mathbf{I}_\mathbf{F}|.$$

By Lemma 3.2, Corollary 3.21, and by Lemma 3.1,

$$\begin{vmatrix} -\rho\mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho\mathbf{F}'\mathbf{F} \end{vmatrix} = |-\rho\mathbf{E}'\mathbf{E}| |-\rho\mathbf{F}'\mathbf{F}| |\mathbf{I} - \mathbf{I}_\mathbf{E}\mathbf{I}_\mathbf{F}/\rho^2|.$$

COROLLARY. The equations $\begin{vmatrix} -\rho\mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho\mathbf{F}'\mathbf{F} \end{vmatrix} = 0$ and $|\rho^2\mathbf{I} - \mathbf{I}_\mathbf{E}\mathbf{I}_\mathbf{F}| = 0$ have the same positive roots, with the same multiplicities.

4. *Projection.* If $\mathfrak{E}$, $\mathfrak{F}$ are complementary spaces, then their bases $\mathbf{E}$, $\mathbf{F}$ join in a regular square matrix $\mathbf{G} = (\mathbf{E}\mathbf{F})$, and any vector $\mathbf{Z}$ has a unique resolution

$$\mathbf{Z} = \mathbf{G}\mathbf{z} = (\mathbf{E}\mathbf{F}) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{E}\mathbf{x} + \mathbf{F}\mathbf{y} = \mathbf{X} + \mathbf{Y},$$

where $\mathbf{z} = \mathbf{G}^{-1}\mathbf{Z}$, into a sum of vectors $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{E}$, $\mathfrak{F}$ obtained from $\mathbf{Z}$ by linear transformations determined just by these spaces. The components $\mathbf{X}$, $\mathbf{Y}$ in this resolution give the *projections* of $\mathbf{Z}$, and the linear transformations which obtain them are the *projectors*, on $\mathfrak{E}$ parallel to $\mathfrak{F}$ and on $\mathfrak{F}$ parallel to $\mathfrak{E}$ respectively. Such a pair of projectors, on and parallel to a pair of complementary spaces, are *complementary projectors*; their two products are null, and their sum is the identity, from which it follows that they are idempotents. Any idempotent represents the projector on its range parallel to its null-space, and if these are orthogonal complements, in which case it must be symmetric, then it is called an *orthogonal projector*; otherwise it is called *oblique*.

If $\mathfrak{E}$, $\mathfrak{F}$ are separate, so that the join $\mathbf{G} = (\mathbf{E}\mathbf{F})$ of their bases $\mathbf{E}$, $\mathbf{F}$ is a base for their union $\mathfrak{G}$, let $\mathfrak{R}$ be a base for the orthogonal complement $\mathfrak{R}$ of $\mathfrak{G}$, which is null just if they are complementary; then $(\mathbf{G}\mathbf{R}) = (\mathbf{E}\mathbf{F}\mathbf{R})$ is a regular square matrix, and there is the unique resolution

$$\begin{aligned} \mathbf{Z} &= \mathbf{G}\mathbf{u} + \mathbf{R}\mathbf{v} = \mathbf{E}\mathbf{x} + \mathbf{F}\mathbf{y} + \mathbf{R}\mathbf{v} \\ &= \mathbf{U} + \mathbf{V} = \mathbf{X} + \mathbf{Y} + \mathbf{V} \end{aligned}$$

of any vector $\mathbf{Z}$ into the sum of its oblique projections $\mathbf{X}$ and $\mathbf{Y}$, on $\mathfrak{E}$ parallel to $\mathfrak{F} \oplus \mathfrak{R}$ and on $\mathfrak{F}$ parallel to $\mathfrak{E} \oplus \mathfrak{R}$, the sum of which is the orthogonal projection $\mathbf{U}$ on $\mathfrak{G}$, and of its orthogonal projection $\mathbf{V}$ on $\mathfrak{R}$. Now, with $\mathbf{I}_\mathbf{F} = \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$, the matrix $\mathbf{J}_\mathbf{F} = \mathbf{I} - \mathbf{I}_\mathbf{F}$ is symmetric idempotent, such that $\mathbf{J}_\mathbf{F}\mathbf{F} = \mathbf{0}$ since $\mathbf{I}_\mathbf{F}\mathbf{F} = \mathbf{F}$, and such that $\mathbf{J}_\mathbf{F}\mathbf{R} = \mathbf{R}$, since $\mathbf{I}_\mathbf{F}\mathbf{R} = \mathbf{0}$, because $\mathfrak{F}$, $\mathfrak{R}$ are orthogonal. Also, since $\mathfrak{E}$, $\mathfrak{F}$ are separate, it follows that $|\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E}| \neq 0$, by Theorem 2.3, Corollary, so that the inverse $(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}$ exists. Accordingly

$$\mathbf{J}_\mathbf{F}\mathbf{Z} = \mathbf{J}_\mathbf{F}\mathbf{E}\mathbf{x} + \mathbf{R}\mathbf{u}, \quad \mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{Z} = \mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E}\mathbf{x},$$

and then $\mathbf{x} = (\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{Z}$, $\mathbf{X} = \mathbf{E}(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{Z}$;

that is, $\mathbf{X} = \mathbf{I}_{\mathbf{E},\mathbf{F}}\mathbf{Z}$, where $\mathbf{I}_{\mathbf{E},\mathbf{F}} = \mathbf{E}(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}\mathbf{E}'\mathbf{J}_\mathbf{F}$.

Hence there are the following conclusions (which are of importance for the theory of partial regression):

THEOREM 4.1. *If $\mathbf{E}, \mathbf{F}$ are bases for separate spaces $\mathfrak{E}, \mathfrak{F}$, then $\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = \mathbf{I}_{\mathbf{E}, \mathbf{F}}$, where $\mathbf{I}_{\mathfrak{E}, \mathfrak{F}}$ is the oblique projector on $\mathfrak{E}$ parallel to the union of $\mathfrak{F}$ with the orthogonal complement of the union of $\mathfrak{E}, \mathfrak{F}$ and $\mathbf{I}_{\mathbf{E}, \mathbf{F}} = \mathbf{E}(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}\mathbf{E}'\mathbf{J}_\mathbf{F}$.*

COROLLARY 4.11. *If $\mathfrak{E}, \mathfrak{F}$ are complementary spaces, then $\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = \mathbf{I}_{\mathbf{E}, \mathbf{F}}$, where $\mathbf{I}_{\mathfrak{E}, \mathfrak{F}}$ is now the oblique projector on $\mathfrak{E}$ parallel to $\mathfrak{F}$.*

COROLLARY 4.12. *If $\mathbf{E}$ is a base for $\mathfrak{E}$, then $\mathbf{I}_{\mathfrak{E}} = \mathbf{I}_{\mathbf{E}}$, where $\mathbf{I}_{\mathfrak{E}}$ is the orthogonal projector on $\mathfrak{E}$ and $\mathbf{I}_{\mathbf{E}} = \mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}\mathbf{E}'$.*

COROLLARY 4.13. *If $\mathbf{E}, \mathbf{F}$ are separate bases with join $\mathbf{G}$, then there is the identity $\mathbf{I}_{\mathbf{G}} = \mathbf{I}_{\mathbf{E}, \mathbf{F}} + \mathbf{I}_{\mathbf{F}, \mathbf{E}}$, and correspondingly for spaces.*

$$\text{Now} \quad \mathbf{I}_{\mathfrak{E}, \mathfrak{F}} + \mathbf{I}_{\mathfrak{F}, \mathfrak{E}} + \mathbf{I}_{\mathfrak{N}} = \mathbf{I},$$

so, since $\mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = \mathbf{I}_{\mathfrak{E}, \mathfrak{F}}$, $\mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{N}} = \mathbf{0}$, and similarly with $\mathfrak{F}$ for $\mathfrak{E}$, it appears that

$$\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} + \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}, \mathfrak{E}} = \mathbf{I}_{\mathfrak{E}}, \quad \mathbf{I}_{\mathfrak{F}, \mathfrak{E}} + \mathbf{I}_{\mathfrak{F}}\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = \mathbf{I}_{\mathfrak{F}},$$

and then, by substitution of $\mathbf{I}_{\mathfrak{F}, \mathfrak{E}}$ from the second relation into the first, that

$$(\mathbf{I} - \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}})\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = \mathbf{I}_{\mathfrak{E}}(\mathbf{I} - \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}}).$$

By Theorem 3.3, Corollary, and by Theorem 4.1, Corollary 4.12, we have $|\mathbf{I} - \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}}| \neq 0$, since $\mathfrak{E}, \mathfrak{F}$ are separate; thus the following theorem is obtained:

THEOREM 4.2. *If $\mathfrak{E}, \mathfrak{F}$ are separate spaces, then the oblique projector $\mathbf{I}_{\mathfrak{E}, \mathfrak{F}}$ on $\mathfrak{E}$ parallel to the union of $\mathfrak{F}$ with the orthogonal complement of the union of $\mathfrak{E}, \mathfrak{F}$ is determined from the orthogonal projectors $\mathbf{I}_{\mathfrak{E}}, \mathbf{I}_{\mathfrak{F}}$ on $\mathfrak{E}, \mathfrak{F}$ by the formula*

$$\mathbf{I}_{\mathfrak{E}, \mathfrak{F}} = (\mathbf{I} - \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}})^{-1}\mathbf{I}_{\mathfrak{E}}(\mathbf{I} - \mathbf{I}_{\mathfrak{E}}\mathbf{I}_{\mathfrak{F}}).$$

COROLLARY. *If $\mathbf{E}, \mathbf{F}$ are separate bases then*

$$\mathbf{E}(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1}\mathbf{E}'\mathbf{J}_\mathbf{F} = (\mathbf{I} - \mathbf{I}_{\mathbf{E}}\mathbf{I}_{\mathbf{F}})^{-1}\mathbf{I}_{\mathbf{E}}(\mathbf{I} - \mathbf{I}_{\mathbf{E}}\mathbf{I}_{\mathbf{F}}).$$

The following theorem can be verified directly:

THEOREM 4.3. *If $(\mathbf{E}\mathbf{F}) = \mathbf{G}$ is a base, then $(\mathbf{G}'\mathbf{G})^{-1} = \mathbf{N}'\mathbf{N}$, $\mathbf{N}'\mathbf{G} = \mathbf{L}$, where*

$$(\mathbf{L}\mathbf{M}) = \mathbf{N}, \quad \mathbf{L} = \mathbf{J}_\mathbf{F}\mathbf{E}(\mathbf{E}'\mathbf{J}_\mathbf{F}\mathbf{E})^{-1} \quad \text{and} \quad \mathbf{M} = \mathbf{J}_\mathbf{E}\mathbf{F}(\mathbf{F}'\mathbf{J}_\mathbf{E}\mathbf{F})^{-1}.$$

COROLLARY 4.31. $\mathbf{G}\mathbf{N}' = \mathbf{E}\mathbf{L}' + \mathbf{F}\mathbf{M}'$, where $\mathbf{I}_{\mathbf{G}} = \mathbf{G}\mathbf{N}'$, $\mathbf{I}_{\mathbf{E}, \mathbf{F}} = \mathbf{E}\mathbf{L}'$, $\mathbf{I}_{\mathbf{F}, \mathbf{E}} = \mathbf{F}\mathbf{M}'$.

COROLLARY 4.32. *If $\mathbf{E}, \mathbf{F}$ are orthogonal, then $\mathbf{L} = \mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}$, $\mathbf{M} = \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}$.*

COROLLARY 4.33. *If $\mathbf{E}, \mathbf{F}$ are complementary, so that $\mathbf{G}$ is a regular square matrix with inverse $\mathbf{N}'$, then $\mathbf{E}\mathbf{L}'$, $\mathbf{F}\mathbf{M}'$ are the complementary oblique projectors on and parallel to the spans of $\mathbf{E}, \mathbf{F}$.*

Spaces $\mathfrak{E}_i$ ($i = 1, \dots, m$) are said to form a *complementary set* if they are mutually separate and their union is the complete space:

$$\mathfrak{E}_i \cap \mathfrak{E}_j = \mathbf{0} \quad (i \neq j), \quad \bigoplus_{i=1}^m \mathfrak{E}_i = \mathfrak{L}.$$

In this case, if $\mathbf{E}_i$ is a base for $\mathfrak{E}_i$ ($i = 1, \dots, m$), then $\mathbf{E} = (\mathbf{E}_1 \dots \mathbf{E}_m)$ is a regular square matrix.

COROLLARY 4.34. *If spaces $\mathfrak{E}_i$ with bases $\mathbf{E}_i$ ($i = 1, \dots, m$) form a complementary set, and if $\mathbf{E} = (\mathbf{E}_1 \dots \mathbf{E}_m)$, $\mathbf{F} = (\mathbf{F}_1 \dots \mathbf{F}_m)$, where $\mathbf{E}^{-1} = \mathbf{F}'$, then $\mathbf{E}_i \mathbf{F}_i$ is the oblique projector on $\mathfrak{E}_i$ parallel to $\mathfrak{F}_i = \bigoplus_{j \neq i} \mathfrak{E}_j$.*

Let $\mathbf{e}, \mathbf{f}$ now denote the orthogonal projectors on spaces $\mathfrak{E}, \mathfrak{F}$ in $\mathfrak{L}$ of dimension p, q and with bases $\mathbf{E}, \mathbf{F}$; so

$$\mathbf{e} = \mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}\mathbf{E}', \quad \mathbf{f} = \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}',$$

and $\bar{\mathbf{e}} = \mathbf{1} - \mathbf{e}$, $\bar{\mathbf{f}} = \mathbf{1} - \mathbf{f}$ are the orthogonal projectors on the orthogonal complements $\bar{\mathfrak{E}} = \mathfrak{L} \ominus \mathfrak{E}$, $\bar{\mathfrak{F}} = \mathfrak{L} \ominus \mathfrak{F}$ of $\mathfrak{E}, \mathfrak{F}$; and $p = \text{rank } \mathbf{e} = \text{trace } \mathbf{e}$, $q = \text{rank } \mathbf{f} = \text{trace } \mathbf{f}$. The numbers $r = \text{rank } \mathbf{e}\mathbf{f}$, $R = \text{trace } \mathbf{e}\mathbf{f}$, such that $R \leq r$, will define the *dimension* and the *coefficient of inclination* between $\mathfrak{E}, \mathfrak{F}$; it appears in § 6 that they are the same if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonally incident, and zero if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonal, also that $\mathfrak{E}$ is completely inclined to $\mathfrak{F}$ just if $r = p$, and included in $\mathfrak{F}$ just if $R = p$. The coefficient of inclination between a pair of vectors $\mathbf{X}, \mathbf{Y}$ or between the rays spanned by those vectors is

$$R = \text{trace } \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}(\mathbf{Y}'\mathbf{Y})^{-1}\mathbf{Y}' = (\mathbf{X}'\mathbf{Y})^2/(\mathbf{X}'\mathbf{X})(\mathbf{Y}'\mathbf{Y}) = \cos^2 \alpha,$$

that is, it is the square of the cosine of the angle α which separates them.

The space $\mathbf{e}\mathfrak{F}$, formed of the orthogonal projections $\mathbf{e}\mathbf{Y}$ in $\mathfrak{E}$ of every vector $\mathbf{Y}$ of $\mathfrak{F}$, gives the orthogonal projection of the space $\mathfrak{F}$ in the space $\mathfrak{E}$. The spaces $\mathbf{e}\mathfrak{F}, \mathbf{f}\mathfrak{E}$ are of the same dimension, given by the dimension of inclination r between $\mathfrak{E}, \mathfrak{F}$. The inclination between a pair of spaces appears equal to the inclination between their orthogonal projections on each other

$$R(\mathfrak{E}, \mathfrak{F}) = R(\mathbf{e}\mathfrak{F}, \mathbf{f}\mathfrak{E}).$$

A pair of spaces $\mathfrak{U}, \mathfrak{V}$ are called *reciprocals* in $\mathfrak{E}, \mathfrak{F}$ if they are the orthogonal projections of each other in $\mathfrak{E}, \mathfrak{F}$:

$$\mathfrak{U} = \mathbf{e}\mathfrak{V}, \quad \mathfrak{V} = \mathbf{f}\mathfrak{U};$$

and a pair of vectors $\mathbf{X}, \mathbf{Y}$ are called *reciprocals* if they span reciprocal rays, in which case

$$\mathbf{X}\lambda = \mathbf{e}\mathbf{Y}, \quad \mathbf{Y}\mu = \mathbf{f}\mathbf{X},$$

where

$$\mathbf{X}'\mathbf{X}\lambda = \mathbf{X}'\mathbf{e}\mathbf{Y} = \mathbf{X}'\mathbf{Y}, \quad \mathbf{Y}'\mathbf{Y}\mu = \mathbf{Y}'\mathbf{f}\mathbf{X} = \mathbf{Y}'\mathbf{X},$$

since $\mathbf{e}\mathbf{X} = \mathbf{X}$, $\mathbf{f}\mathbf{Y} = \mathbf{Y}$, because $\mathbf{X}, \mathbf{Y}$ belong to $\mathfrak{E}, \mathfrak{F}$, since $\lambda, \mu \neq 0$; and the numbers

$$\lambda = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}, \quad \mu = (\mathbf{Y}'\mathbf{Y})^{-1}\mathbf{Y}'\mathbf{X}$$

are such that

$$\lambda\mu = \rho^2,$$

where

$$\rho^2 = (\mathbf{X}'\mathbf{Y})^2/(\mathbf{X}'\mathbf{X})(\mathbf{Y}'\mathbf{Y}) = \cos^2 \alpha$$

is again the coefficient of inclination between the vectors, or the rays which they span. Now it follows that

$$\mathbf{e}\mathbf{f}\mathbf{X} = \mathbf{e}\mathbf{Y}\mu = \mathbf{X}\lambda\mu = \mathbf{X}\rho^2, \quad \mathbf{f}\mathbf{e}\mathbf{Y} = \mathbf{f}\mathbf{X}\lambda = \mathbf{Y}\mu\lambda = \mathbf{Y}\rho^2,$$

so that $\mathbf{X}, \mathbf{Y}$ are latent vectors of $\mathbf{ef}, \mathbf{fe}$ with characteristic value ρ^2 . Also, if $\mathbf{X}$ is now any latent vector of $\mathbf{ef}$ for a positive characteristic value ρ^2 , so that $\mathbf{efX} = \mathbf{X}\rho^2$, and if now $\mathbf{Y} = \mathbf{fX}$, then $\mathbf{eY} = \mathbf{efX} = \mathbf{X}\rho^2$, and $\mathbf{feY} = \mathbf{fX}\rho^2 = \mathbf{Y}\rho^2$, so that $\mathbf{Y}$ is a latent vector of $\mathbf{fe}$ with characteristic value ρ^2 , and $\mathbf{X}, \mathbf{Y}$ are reciprocal vectors in $\mathfrak{E}, \mathfrak{F}$ with inclination coefficient ρ^2 :

THEOREM 4.4. *Any reciprocal vectors $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{E}, \mathfrak{F}$ with inclination ρ^2 are latent vectors of $\mathbf{ef}, \mathbf{fe}$ both with characteristic value ρ^2 ; and if $\mathbf{X}$ is any latent vector of $\mathbf{ef}$ for a positive characteristic value ρ^2 then $\mathbf{Y} = \mathbf{fX}$ is a latent vector of $\mathbf{fe}$ for the same characteristic value, and $\mathbf{X}, \mathbf{Y}$ are reciprocal vectors in $\mathfrak{E}, \mathfrak{F}$ with inclination ρ^2 .*

The next theorem establishes a procedure by which the intersection of a pair of spaces can be calculated when the bases are given.

THEOREM 4.5. *The intersection of the space $\mathfrak{E}, \mathfrak{F}$ is the null-space of the matrix $\mathbf{1} - \mathbf{ef}$.*

Let $\mathfrak{J}, \mathfrak{N}$ denote the intersection and null-space considered. It is evident that $\mathfrak{J} \subset \mathfrak{N}$. Let $\mathbf{X}$ be any vector of $\mathfrak{N}$, so that $\mathbf{efX} = \mathbf{X}$, and let $\mathbf{Y} = \mathbf{fX}$, so that $\mathbf{X} = \mathbf{eY}$. Then $\mathbf{X}, \mathbf{Y}$ belong to the ranges $\mathfrak{E}, \mathfrak{F}$ of $\mathbf{e}, \mathbf{f}$; and

$$\mathbf{X}'\mathbf{X} = \mathbf{X}'\mathbf{eY} = \mathbf{X}'\mathbf{Y} = \mathbf{X}'\mathbf{fY} = \mathbf{Y}'\mathbf{Y}.$$

Therefore, by Schwartz's inequality, $\mathbf{X} = \mathbf{Y}$, so that any vector $\mathbf{X}$ of $\mathfrak{N}$ belongs to both $\mathfrak{E}$ and $\mathfrak{F}$, that is, to $\mathfrak{J}$, and hence $\mathfrak{N} \subset \mathfrak{J}$. Accordingly, $\mathfrak{J} = \mathfrak{N}$.

COROLLARY. *The space $\mathfrak{E} \ominus \mathfrak{F}$ is the null-space of the matrix $\mathbf{1} - \mathbf{ef}$.*

THEOREM 4.6. $\mathfrak{E} \ominus \mathbf{e}\mathfrak{F} = \mathfrak{E} \ominus \mathfrak{F} = \mathfrak{E} \ominus \mathbf{f}\mathfrak{E}.$

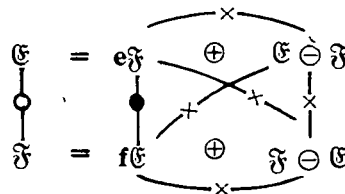
That is, the orthogonal complements in $\mathfrak{E}$ of $\mathbf{e}\mathfrak{F}, \mathfrak{F}$ and $\mathbf{f}\mathfrak{E}$ are the same. If $\mathbf{U} = \mathbf{eV}$, then $\mathbf{eU} = \mathbf{U}$, and hence $\mathbf{U} = \mathbf{eV}$ and $\mathbf{feU} = \mathbf{0}$ is equivalent to $\mathbf{U} = \mathbf{eV}$ and $\mathbf{fU} = \mathbf{0}$; therefore $\mathfrak{E} \ominus \mathbf{e}\mathfrak{F} = \mathfrak{E} \ominus \mathfrak{F}$. Also, if $\mathbf{eU} = \mathbf{U}$, then $\mathbf{fU} = \mathbf{0}$ is equivalent to $\mathbf{feU} = \mathbf{0}$; moreover, $\mathbf{0} = \mathbf{efU} = (\mathbf{fe})'\mathbf{feU}$ implies $\mathbf{feU} = \mathbf{0}$. Thus $\mathbf{U} = \mathbf{eV}$ and $\mathbf{efU} = \mathbf{0}$ is equivalent to $\mathbf{U} = \mathbf{eV}$ and $\mathbf{fU} = \mathbf{0}$; therefore $\mathfrak{E} \ominus \mathfrak{F} = \mathfrak{E} \ominus \mathbf{f}\mathfrak{E}$.

THEOREM 4.7. *The spaces $\mathbf{e}\mathfrak{F}, \mathbf{f}\mathfrak{E}$ are reciprocals in $\mathfrak{E}, \mathfrak{F}$ and they are totally inclined.*

For the matrices $\mathbf{ef}, \mathbf{efe} = \mathbf{ef}(\mathbf{ef})'$ have the same range. Now every vector in one of the pair is inclined to some vector of the other, so they are totally inclined.

Let the signs $\circ, \bullet$ and $\times$ between a pair of spaces indicate that they are inclined, totally inclined and orthogonal. Theorems 4.6 and 4.7 show the following scheme of decomposition, and Theorem 2.5, Corollary, shows the procedure by which it can be effected.

THEOREM 4.8. *Any pair of inclined spaces $\mathfrak{E}, \mathfrak{F}$ have decomposition in the scheme:*



5. *Rank, multiplicity and orthogonality.* The theory of the latent vectors and characteristic values of products of orthogonal projectors is required for the subsequent theory of the canonical decomposition of a pair of spaces relative to each other, and the construction of bases in each which form a canonical pair. Since $\mathbf{e}, \mathbf{f}$ are symmetric matrices, their products $\mathbf{ef}, \mathbf{fe}$ are transposes of each other, so they have exactly the same characteristic values, with the same multiplicities; and these values and multiplicities characterize a symmetric relation between the spaces $\mathfrak{E}, \mathfrak{F}$ or are characteristic of the configuration the spaces present relative to each other.

THEOREM 5.1. *The characteristic values of $\mathbf{ef}$ are all real, at least zero and at most unity.*

For the proof, see Afriat (3).

Let m_λ denote the multiplicity of any value λ as a characteristic value of $\mathbf{ef}$. Then any λ for which $m_\lambda \neq 0$ may be called a characteristic value for the spaces $\mathfrak{E}, \mathfrak{F}$ with this multiplicity. Since $0 \leq \lambda \leq 1$ if $m_\lambda \neq 0$, by Theorem 4.1, positive acute angles α are determined by $\lambda = \cos^2 \alpha$. The values λ ($m_\lambda \neq 0$) are to define the *canonical coefficients of inclination* between the spaces, and the corresponding angles α define the *canonical angles*. If the canonical angles between a pair of spaces are all the same, the spaces are called *isogonal*.

Let $\mathfrak{E}_\lambda, \mathfrak{F}_\lambda$ denote the null-spaces of $\lambda \mathbf{1} - \mathbf{ef}, \lambda \mathbf{1} - \mathbf{fe}$. Then $\mathfrak{E}_\lambda \neq 0, \mathfrak{F}_\lambda \neq 0$ just if $m_\lambda \neq 0$, and $\mathfrak{E}_\lambda \subset \mathbf{e}\mathfrak{F}, \mathfrak{F}_\lambda \subset \mathbf{f}\mathfrak{E}$ if $\lambda \neq 0$, in which case they are to define a *canonical pair of subspaces* of $\mathfrak{E}, \mathfrak{F}$. The next theorem shows that the dimensions of $\mathfrak{E}_\lambda, \mathfrak{F}_\lambda$ are equal to m_λ .

THEOREM 5.2. *The nullity of the matrix $\lambda \mathbf{1} - \mathbf{ef}$ is equal to the multiplicity of λ as a root of the equation $|\lambda \mathbf{1} - \mathbf{ef}| = 0$.*

Proof is given in Afriat (3).

COROLLARY 5.21. *The rank of $\mathbf{ef}$ is the total multiplicity of its non-zero characteristic values.*

That is, $r = \sum_{\lambda \neq 0} m_\lambda$, where r is the dimension of inclination between $\mathfrak{E}, \mathfrak{F}$.

THEOREM 5.3. $|\mathbf{1} - \mathbf{ef}| = \prod_{\lambda} (1 - \lambda)^{m_\lambda}, \quad |\mathbf{1} - \mathbf{ef}| = 0^{p-r} \prod_{\lambda \neq 0} \lambda^{m_\lambda}.$

For the characteristic values of $\mathbf{1} - \mathbf{ef}$ are $1 - \lambda$ with multiplicity m_λ , and the determinant of a matrix is the product of its characteristic values; which proves the first part of the theorem. To prove the second part, it is seen, by Theorem 3.3, that

$$|(\mathbf{E}'\mathbf{E})^{-1} \mathbf{E}'\mathbf{f}\mathbf{E}| = |\mathbf{E}'\mathbf{f}\mathbf{E}| / |\mathbf{E}'\mathbf{E}| = |\mathbf{1} - \mathbf{ef}|.$$

But, by Lemma 3.1, Corollary 3.12, the matrices

$$(\mathbf{E}'\mathbf{E})^{-1} \mathbf{E}'\mathbf{f}\mathbf{E}, \quad \mathbf{ef}$$

have the same non-zero characteristic values, with the same multiplicities; and now the required conclusion follows.

The reciprocal vectors of a pair of spaces, which by Theorem 3.4 are the same as the reciprocal latent vectors of the products of the orthogonal projectors on the spaces, have the following properties of orthogonality:

THEOREM 5.4. *If $\mathbf{U}_\lambda, \mathbf{V}_\lambda$ is a pair of reciprocal vectors of $\mathfrak{E}, \mathfrak{F}$ with inclination λ , for $\lambda = \alpha, \beta \neq 0$, then the orthogonality conditions $\mathbf{U}'_\alpha \mathbf{U}_\beta = 0, \mathbf{U}'_\alpha \mathbf{V}_\beta = 0, \mathbf{V}'_\alpha \mathbf{U}_\beta = 0$ and $\mathbf{V}'_\alpha \mathbf{V}_\beta = 0$ are equivalent, and they hold automatically if $\alpha \neq \beta$.*

Suppose they are unit vectors, so that $(\mathbf{U}'_\lambda \mathbf{V}_\lambda)^2 = \lambda$, and

$$\mathbf{eV}_\lambda = \mathbf{U}_\lambda \lambda^{\frac{1}{2}}, \quad \mathbf{fU}_\lambda = \mathbf{V}_\lambda \lambda^{\frac{1}{2}}.$$

Then

$$\mathbf{U}'_\mu \mathbf{V}_\lambda = \mathbf{U}'_\mu \mathbf{eV}_\lambda = \mathbf{U}'_\mu \mathbf{U}_\lambda \lambda^{\frac{1}{2}},$$

since $\mathbf{eU}_\mu = \mathbf{U}_\mu, \mu = \alpha, \beta \neq 0$. Hence $\mathbf{U}'_\mu \mathbf{V}_\lambda = 0$ is equivalent to $\mathbf{U}'_\mu \mathbf{U}_\lambda = 0$, since $\lambda \neq 0$, and the remaining equivalences follow by symmetry. Further,

$$\alpha^{\frac{1}{2}} \mathbf{V}'_\alpha \mathbf{eV}_\beta = \mathbf{U}'_\alpha \mathbf{fU}_\beta \beta^{\frac{1}{2}}, \quad \alpha \mathbf{U}'_\alpha \mathbf{V}_\beta = \mathbf{U}'_\alpha \mathbf{V}_\beta \beta,$$

so that $\mathbf{U}'_\alpha \mathbf{V}_\beta = 0$ if $\alpha \neq \beta$.

6. *Configuration scales.* A set of two or, in principle, of three or more subspaces of a given space is considered as forming a configuration of subspaces; and a configuration is viewed as attaining to or departing from some different model types, according to measurement in different scales, each of which measures a maximum just when a model is perfectly attained and measures zero when another model, which defines the opposite, is attained. Thus, for a configuration of two spaces, there are the following ideal models: the spaces are orthogonal, orthogonally incident, isogonal, identical, incident, or one space is included in another. Also, for a configuration of two spaces, there are the following models: the three spaces are mutually orthogonal, a pair are identical, or a pair are orthogonal and the third is their union. Coefficients are to be defined which set any configuration in a scale, in respect to each model. They present features of the general relation the spaces have to each other which it is possible to translate into statistical conceptions which are fitting in the general subject of multivariate analysis.

LEMMA 6.1. (i) *For any matrices $\mathbf{a}_i$ of the same order,*

$$(\text{trace } \mathbf{a}'_0 \mathbf{a}_1)^2 \leq (\text{trace } \mathbf{a}'_0 \mathbf{a}_0) (\text{trace } \mathbf{a}'_1 \mathbf{a}_1),$$

and the equality holds if and only if $\lambda_0 \mathbf{a}_0 + \lambda_1 \mathbf{a}_1 = \mathbf{0}$ for some scalars λ_0, λ_1 which are not both zero.

$$(ii) \quad \prod_0^2 (\text{trace } \mathbf{a}'_i \mathbf{a}_i) + 2 \prod_0^2 (\text{trace } \mathbf{a}'_j \mathbf{a}_k) \geq \sum_0^2 (\text{trace } \mathbf{a}'_i \mathbf{a}_i) (\text{trace } \mathbf{a}'_j \mathbf{a}_k)^2,$$

and the equality holds if and only if $\sum_0^2 \lambda_i \mathbf{a}_i = \mathbf{0}$ for some scalars λ_i which are not all zero.

If the matrices $\mathbf{a}_i$ are of order $m \times n$, let $\mathbf{A}_i$ denote the vector of order mn whose elements are the elements of $\mathbf{A}_i$ ordered by their position. In the first instance let $\mathbf{K} = (\mathbf{A}_1 \mathbf{A}_2)$, and in the second let $\mathbf{K} = (\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3)$. Then $|\mathbf{K}' \mathbf{K}| \geq 0$, and the equality holds just if $\mathbf{K} \boldsymbol{\lambda} = \mathbf{0}$ for some non-null vector $\boldsymbol{\lambda}$, in the first instance of order two and in the second of order three. The lemma is a restatement of this result.

LEMMA 6.2. (i) *Two symmetric idempotents are linearly dependent if and only if they are identical.*

(ii) *Three symmetric idempotents are linearly dependent if and only if either a pair are identical or a pair are orthogonal and the third is their sum.*

The proof of this lemma is long but straightforward, and it will be taken for granted.

LEMMA 6.3. For any symmetric idempotents $\mathbf{e}_i$ ($i = 0, 1, \dots$)

$$(i) \quad (\text{trace } \mathbf{e}_0 \mathbf{e}_1)^2 \leq (\text{trace } \mathbf{e}_0) (\text{trace } \mathbf{e}_1),$$

where equality holds if and only if $\mathbf{e}_0 = \mathbf{e}_1$;

$$(ii) \quad \prod_0^2 (\text{trace } \mathbf{e}_i \mathbf{e}_i) + 2 \prod_0^2 (\text{trace } \mathbf{e}_j \mathbf{e}_k) \geq \sum_0^2 (\text{trace } \mathbf{e}_i) (\text{trace } \mathbf{e}_j \mathbf{e}_k)^2,$$

where equality holds if and only if either $\mathbf{e}_j = \mathbf{e}_k$ or $\mathbf{e}_j \mathbf{e}_k = \mathbf{0}$ and $\mathbf{e}_i = \mathbf{e}_j + \mathbf{e}_k$ for some i, j, k which are a permutation of $0, 1, 2$.

This is a corollary of Lemmas 5.1 and 5.2.

Since $\mathbf{e}, \mathbf{f}$ are symmetric idempotents, and the trace of a product of matrices depends only on the cyclical order of the factors,

$$R = \text{trace } \mathbf{e} \mathbf{f} = \text{trace } (\mathbf{e} \mathbf{f})' \mathbf{e} \mathbf{f} \geq 0,$$

and the equality holds just if $\mathbf{e} \mathbf{f} = \mathbf{0}$, which is the case just if $\mathfrak{E}, \mathfrak{F}$ are orthogonal; also, since $p = \text{trace } \mathbf{e}$,

$$p - R = \text{trace } \mathbf{e} - \text{trace } \mathbf{e} \mathbf{f} = \text{trace } \mathbf{e} \bar{\mathbf{f}} \geq 0,$$

and equality holds if and only if $\mathbf{e} \bar{\mathbf{f}} = \mathbf{0}$, that is, $\mathbf{e} \mathbf{f} = \mathbf{e}$, which is the case just if $\mathfrak{E} \subset \mathfrak{F}$. Moreover, if $R = r$, then all the non-zero characteristic values of $\mathbf{e} \mathbf{f}$, with total multiplicity r , are equal to unity. It can be shown then that $\mathbf{e}, \mathbf{f}$ commute, so that $\mathfrak{E}, \mathfrak{F}$ are orthogonally incident. Accordingly the number R , defined for any pair of spaces $\mathfrak{E}, \mathfrak{F}$ and called the *coefficient of inclination* between them, has the following properties:

THEOREM 6.4. $0 \leq R \leq p, q, r$ for any spaces $\mathfrak{E}, \mathfrak{F}$ of dimension p, q and dimension of inclination r ; $R = 0$ if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonal; $R = r$ if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonally incident; $R = p$ if and only if $\mathfrak{E}$ is included in $\mathfrak{F}$.

Thus there may be defined the positive numbers

$$\text{Cos } (\mathfrak{E}, \mathfrak{F}) = (\text{trace } \mathbf{e} \mathbf{f})^{\frac{1}{2}}, \quad \text{Sin } (\mathfrak{E}, \mathfrak{F}) = (\text{trace } \mathbf{e} \bar{\mathbf{f}})^{\frac{1}{2}},$$

and

$$\text{Tan } (\mathfrak{E}, \mathfrak{F}) = \text{Sin } (\mathfrak{E}, \mathfrak{F}) / \text{Cos } (\mathfrak{E}, \mathfrak{F}),$$

such that

$$\text{Sin}^2 (\mathfrak{E}, \mathfrak{F}) + \text{Cos}^2 (\mathfrak{E}, \mathfrak{F}) = p,$$

called the *additive cosine, sine and tangent* of $\mathfrak{E}$ in $\mathfrak{F}$, which are indices of the relation of inclusion of the space $\mathfrak{E}$ in the space $\mathfrak{F}$. The sine and the tangent are not symmetrical functions of the spaces; but the cosine, which is to say also the coefficient of inclination, is such a function.

In terms of the canonical angles between the spaces,

$$\text{Cos}^2 (\mathfrak{E}, \mathfrak{F}) = \sum_{\alpha} m_{\alpha} \cos^2 \alpha, \quad \text{Sin}^2 (\mathfrak{E}, \mathfrak{F}) = p - r + \sum_{\alpha} m_{\alpha} \sin^2 \alpha.$$

If $\mathfrak{E}, \mathfrak{F}$ are orthogonal, by their union, and $\mathfrak{H}$ another space, then

$$\text{Cos}^2 (\mathfrak{G}, \mathfrak{H}) = \text{Cos}^2 (\mathfrak{E}, \mathfrak{H}) + \text{Cos}^2 (\mathfrak{F}, \mathfrak{H});$$

for then $\mathbf{g} = \mathbf{e} + \mathbf{f}$ is the orthogonal projector of $\mathfrak{G}$, so that

$$\text{trace } \mathbf{g} \mathbf{h} = \text{trace } \mathbf{e} \mathbf{h} + \text{trace } \mathbf{f} \mathbf{h}.$$

For example, let l, m, n be three independent rays, concurrent on the origin O ; and let π be the plane through m, n ; and let l' be the orthogonal projection of l on π . Then the inclination of l to π is equal to the inclination of l to l' , and this is the cosine of the angle between l and l' , say α . If β, γ are the angles between l and m, l and n , it follows that if m, n are orthogonal then $\cos^2 \alpha = \cos^2 \beta + \cos^2 \gamma$.

It follows from Lemma 6.3 (i) that there is defined for any pair of spaces $\mathfrak{E}, \mathfrak{F}$ a non-negative real number Q , to be called their *coefficient of disinclination*, the square of which is given by

$$Q^2 = pq - R^2 \geq 0,$$

where equality holds just if $\mathfrak{E}, \mathfrak{F}$ are identical, and where $Q^2 = pq$ just if $R^2 = 0$, which, by Theorem 5.4, is the case just if $\mathfrak{E}, \mathfrak{F}$ are orthogonal. Like the coefficient of inclination, the coefficient of disinclination is a symmetrical function of the two spaces, and it has the following properties:

THEOREM 6.5. $0 \leq Q^2 \leq pq$ for any spaces $\mathfrak{E}, \mathfrak{F}$ of dimension p, q ; $Q^2 = 0$ if and only if $\mathfrak{E}, \mathfrak{F}$ are identical; $Q^2 = pq$ if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonal.

Now let

$$S = |1 - \mathbf{ef}|;$$

then it appears, from Theorems 5.1 and 5.3, that $0 \leq S \leq 1$, and that $S = 1$ just if $\mathfrak{E}, \mathfrak{F}$ are orthogonal; and, by Theorem 4.5, $S = 0$ if and only if $\mathfrak{E}, \mathfrak{F}$ are incident. Accordingly the number S , defined for any pair of spaces, which will be called the *coefficient of separation* between them, has the following properties:

THEOREM 6.6. $0 \leq S \leq 1$ for any spaces $\mathfrak{E}, \mathfrak{F}$; $S = 0$ if and only if $\mathfrak{E}, \mathfrak{F}$ are incident; $S = 1$ if and only if $\mathfrak{E}, \mathfrak{F}$ are orthogonal.

The number

$$C = |1 - \mathbf{ef}|$$

may be called the *coefficient of inclusion* of $\mathfrak{E}$ in $\mathfrak{F}$, and its properties now appear as a corollary of Theorem 6.6.

THEOREM 6.7. $0 \leq C \leq 1$ for any spaces $\mathfrak{E}, \mathfrak{F}$; $C = 0$ if and only if $\mathfrak{E}$ is incompletely inclined to $\mathfrak{F}$; $C = 1$ if and only if $\mathfrak{E}$ is included in $\mathfrak{F}$.

It follows that there are defined the non-negative real numbers

$$\sin(\mathfrak{E}, \mathfrak{F}) = |1 - \mathbf{ef}|^{\frac{1}{2}}, \quad \cos(\mathfrak{E}, \mathfrak{F}) = |1 - \mathbf{ef}|^{\frac{1}{2}},$$

and

$$\tan(\mathfrak{E}, \mathfrak{F}) = \sin(\mathfrak{E}, \mathfrak{F}) / \cos(\mathfrak{E}, \mathfrak{F}),$$

which may be called the *multiplicative sine, cosine and tangent* of $\mathfrak{E}$ on $\mathfrak{F}$. In terms of the canonical angles α between the spaces with multiplicities m_α , it follows from Theorem 5.3 that

$$\sin(\mathfrak{E}, \mathfrak{F}) = \prod_{\alpha} \sin^{m_\alpha} \alpha, \quad \cos(\mathfrak{E}, \mathfrak{F}) = 0^{p-r} \prod_{\alpha \neq \frac{1}{2}\pi} \cos^{m_\alpha} \alpha,$$

and

$$\tan(\mathfrak{E}, \mathfrak{F}) = 0^{r-p} \prod_{\alpha \neq \frac{1}{2}\pi} \tan^{m_\alpha} \alpha.$$

Their properties follow from Theorems 6.6 and 6.7.

It is noted that, if $\mathfrak{E}, \mathfrak{F}$ are of unequal dimension, then at least one of the coefficients of inclusion $\cos^2(\mathfrak{E}, \mathfrak{F}), \cos^2(\mathfrak{F}, \mathfrak{G})$ is zero; and if they are of equal dimension, then the coefficients of inclusion are identical. The coefficient of separation, however, is a symmetrical function of the two spaces.

Let R_{ij}, Q_{ij} denote the coefficients of inclination and disinclination between spaces $\mathfrak{E}_i, \mathfrak{E}_j$ so that, if $\mathbf{e}_i$ is the orthogonal projector on $\mathfrak{E}_i$ and p_i the dimension, then

$$p_i = \text{trace } \mathbf{e}_i, \quad R_{ij} = \text{trace } \mathbf{e}_i \mathbf{e}_j \quad \text{and} \quad Q_{ij}^2 = p_i p_j - R_{ij}^2.$$

Now for three spaces i, j, k having indices i, j, k define

$$Q_{ijk}^3 = p_i p_j p_k + 2R_{jk} R_{ki} R_{ij} - p_i R_{jk}^2 - p_j R_{ki}^2 - p_k R_{ij}^2,$$

and call the real number Q_{ijk} with cube thus defined the *coefficient of dissociation* between the three spaces. Then the coefficient of dissociation satisfies the following relation with the coefficients of inclination and disinclination between the pairs:

THEOREM 6.8. $p_i Q_{ijk}^3 = Q_{ij}^2 Q_{ik}^2 - (p_i R_{jk} - R_{ij} R_{ik})^2.$

For, from the definition,

$$p_i Q_{ijk}^3 = (p_i p_j - R_{ij}^2)(p_i p_k - R_{ik}^2) - (p_i R_{jk} - R_{ij} R_{ik})^2.$$

THEOREM 6.9. (i) $0 \leq p_i Q_{ijk}^3 \leq Q_{ij}^2 Q_{ik}^2 \leq p_i^2 p_j p_k$ for any three spaces;

(ii) $Q_{ijk}^3 = 0$ if and only if either a pair of the spaces are identical, or a pair are orthogonal and the third is identical with their union;

(iii) $Q_{ijk}^3 = p_i p_j p_k$ if and only if the three spaces are orthogonal to each other.

By Theorem 6.3, $Q_{ijk}^3 \geq 0$, and equality holds just under the conditions stated. By Theorem 6.8, the maximum of Q_{ijk}^3 is attained when Q_{ij}, Q_{ik} are maximum and $(p_i R_{jk} - R_{ij} R_{ik})^2$ is minimum, that is, zero, and this is precisely when the spaces are all orthogonal.

THEOREM 6.10.

$$R_{ij} R_{ik} - Q_{ij} Q_{ik} \leq p_i R_{jk} \leq R_{ij} R_{ik} + Q_{ij} Q_{ik}$$

and an equality holds if and only if either a pair of the spaces are identical or a pair are orthogonal and the third is identical with their union.

This result follows by Theorems 5.8 and 5.9, and gives limits for the inclination between two spaces when their inclinations and disinclinations to a third are given.

Thus if two spaces have high inclinations and low disinclinations to a third, they must have high inclination to each other. Theorem 6.10 has a similar form to, though it is not precisely a generalization of, the following result. Let l, m, n be three rays, let m, n make angles β, γ with l and let α be the angle they make together. Then

$$\cos \beta \cos \gamma - \sin \beta \sin \gamma \leq \cos \alpha \leq \cos \beta \cos \gamma + \sin \beta \sin \gamma,$$

and an equality is attained just when l lies in the plane through m, n , since then $\alpha = \beta + \gamma$ or $\alpha = \beta - \gamma$.

7. *Volume.* A base $\mathbf{E}$ with span $\mathfrak{E}$ is composed of p independent vectors $\mathbf{E}_1, \dots, \mathbf{E}_p$ and the p -dimensional region in $\mathfrak{E}$ of vectors

$$\mathbf{X} = \mathbf{E}_1 \mathbf{x}_1 + \dots + \mathbf{E}_p \mathbf{x}_p = \mathbf{E} \mathbf{x} \quad (0 \leq x_i \leq 1, i = 1, \dots, p)$$

defines the parallelotope $\mathcal{P} = [\mathbf{E}]$ determined by the vectors or by the base, with volume denoted by $|\mathcal{P}|$.

If $\mathbf{E}, \mathbf{F}$ are separate bases of dimension p, q and $\mathcal{P} = [\mathbf{E}]$, $\mathcal{Q} = [\mathbf{F}]$, then the parallelotope of dimension $p + q$ determined by their join $[\mathbf{EF}]$ is called the product $\mathcal{P} \times \mathcal{Q}$ of $\mathcal{P}, \mathcal{Q}$.

THEOREM 7.1. $|\mathcal{P}|^2 = |\mathbf{E}'\mathbf{E}|$.

Proof is by induction on the dimension p of the parallelotope. The theorem is verified for dimension $p = 1$, since the volume of a single vector is simply its length. It will be proved for dimension $p = r$ on the hypothesis of its validity for dimension $p = r - 1$. Thus, with $\mathcal{P} = [\mathbf{E}]$, let $\mathcal{Q} = [\mathbf{F}]$ where $\mathbf{F} = (\mathbf{E}_1 \dots \mathbf{E}_{r-1})$, so that $\mathbf{E} = (\mathbf{F}\mathbf{E}_r)$, and let $\mathcal{P}_r = [\mathbf{E}_r]$, so that $\mathcal{P} = \mathcal{Q} \times \mathcal{P}_r$. Now $|\mathcal{P}| = |\mathcal{Q}| p_r$, where p_r is the length of the perpendicular from the extremity of $\mathbf{E}_r$ onto the space $\mathfrak{E}$ containing $\mathcal{P}$; and this perpendicular is given by the vector $(\mathbf{1} - \mathbf{f})\mathbf{E}_r$, where $\mathbf{f} = \mathbf{F}(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'$ obtained by taking the orthogonal projection of $\mathbf{E}_r$ on the orthogonal complement of $\mathfrak{F}$, so that $p_r^2 = \mathbf{E}_r'(\mathbf{1} - \mathbf{f})\mathbf{E}_r$. Let $\mathbf{e}_r = \mathbf{E}_r(\mathbf{E}_r'\mathbf{E}_r)^{-1}\mathbf{E}_r'$. Then

$$\mathbf{E}_r'(\mathbf{1} - \mathbf{f})\mathbf{E}_r = |\mathbf{E}_r'\mathbf{E}_r| |\mathbf{1} - \mathbf{f}\mathbf{e}_r|$$

by Theorem 4.3. Accordingly, by the inductive hypothesis $|\mathcal{Q}|^2 = |\mathbf{F}'\mathbf{F}|$, and by Theorem 4.3 again,

$$|\mathcal{P}|^2 = |\mathcal{Q}|^2 p_r^2 = |\mathbf{F}'\mathbf{F}| |\mathbf{E}_r'\mathbf{E}_r| |\mathbf{1} - \mathbf{f}\mathbf{e}_r| = |\mathbf{E}'\mathbf{E}|.$$

COROLLARY. $|\mathcal{P} \times \mathcal{Q}| = |\mathcal{P}| |\mathcal{Q}| \sin(\mathcal{P}, \mathcal{Q})$.

For, if $\mathcal{P} = [\mathbf{E}]$, $\mathcal{Q} = [\mathbf{F}]$, then $|\mathcal{P} \times \mathcal{Q}| = |[\mathbf{EF}]| = |[\mathbf{G}]|$; and

$$|\mathbf{G}'\mathbf{G}| = |\mathbf{E}'\mathbf{E}| |\mathbf{F}'\mathbf{F}| |\mathbf{1} - \mathbf{e}\mathbf{f}|.$$

8. Orthogonalization. If a base $\mathbf{E}$, composed of p independent vectors $\mathbf{E}_1, \dots, \mathbf{E}_p$, is given, then an equivalent orthonormal base $\mathbf{U}$, composed of p orthogonal unit vectors $\mathbf{U}_1, \dots, \mathbf{U}_p$, can be constructed with each $\mathbf{U}_r$ in the span of $\mathbf{E}_1, \dots, \mathbf{E}_r$ ($r = 1, \dots, p$). For, suppose the construction can be carried out for $\mathbf{U}_r$ ($r = 1, \dots, m - 1$). Let $\mathbf{F} = (\mathbf{E}_1 \dots \mathbf{E}_{m-1})$, $\mathbf{V} = (\mathbf{U}_1 \dots \mathbf{U}_{m-1})$, so that since, by hypothesis, the bases $\mathbf{F}, \mathbf{V}$ are equivalent, $\mathbf{I}_{\mathbf{F}} = \mathbf{I}_{m-1} = \mathbf{I}_{\mathbf{V}} = \mathbf{V}\mathbf{V}'$; and let $\mathbf{J}_{m-1} = \mathbf{I} - \mathbf{I}_{m-1}$. Then the vector $\mathbf{U}_m = (\mathbf{E}_m' \mathbf{J}_{m-1} \mathbf{E}_m)^{-\frac{1}{2}} \mathbf{J}_{m-1} \mathbf{E}_m$ is a unit vector orthogonal to $\mathbf{U}_1, \dots, \mathbf{U}_{m-1}$ such that $\mathbf{E}_1, \dots, \mathbf{E}_m$ and $\mathbf{U}_1, \dots, \mathbf{U}_m$ have the same span. Thus the construction can be carried m steps if it can be carried $m - 1$; therefore, since its possibility is evident for $m = 1$, it follows by induction that it is possible for $m = p$.

THEOREM 8.1. If $\mathbf{E}_1, \dots, \mathbf{E}_p$ are p independent vectors, $\mathbf{J}_{m-1}$ the orthogonal projector on the orthogonal complement of the span of $\mathbf{E}_1, \dots, \mathbf{E}_{m-1}$, and

$$\mathbf{U}_m = (\mathbf{E}_m' \mathbf{J}_{m-1} \mathbf{E}_m)^{-\frac{1}{2}} \mathbf{J}_{m-1} \mathbf{E}_m \quad (m = 1, \dots, p),$$

then $\mathbf{U}_1, \dots, \mathbf{U}_p$ are p orthogonal unit vectors such that $\mathbf{E}_1, \dots, \mathbf{E}_m$ and $\mathbf{U}_1, \dots, \mathbf{U}_m$ have the same span ($m = 1, \dots, p$).

This theorem gives explicit formulae for the Gramm-Schmidt orthogonalization process.

9. Canonical decomposition.

THEOREM 9.1. If $\mathfrak{E}_\lambda, \mathfrak{F}_\lambda$ are the null-spaces of $\lambda 1 - \mathbf{ef}$, $\lambda 1 - \mathbf{fe}$ ($\lambda \neq 0$), and $\mathfrak{E}_0, \mathfrak{F}_0$ are the null-spaces of $1 - \mathbf{ef}$, $1 - \mathbf{fe}$, then

$$\mathfrak{E} = \bigoplus_{\lambda} \mathfrak{E}_{\lambda}, \quad \mathfrak{F} = \bigoplus_{\lambda} \mathfrak{F}_{\lambda},$$

where the spaces $\mathfrak{E}_1, \mathfrak{F}_1$ are identical, the spaces $\mathfrak{E}_{\lambda}, \mathfrak{F}_{\lambda}$ ($\lambda \neq 0$) are isogonal, and the spaces $\mathfrak{E}_{\lambda}, \mathfrak{E}_{\mu}; \mathfrak{E}_{\lambda}, \mathfrak{F}_{\mu}; \mathfrak{F}_{\lambda}, \mathfrak{F}_{\mu}$ ($\lambda \neq \mu$) and $\mathfrak{E}_0, \mathfrak{F}_0$ are orthogonal.

Also
$$\mathfrak{E}_{\lambda} = \bigoplus_{i=1}^{m_{\lambda}} \mathfrak{E}_{\lambda,i}, \quad \mathfrak{F}_{\lambda} = \bigoplus_{i=1}^{m_{\lambda}} \mathfrak{F}_{\lambda,i} \quad (\lambda \neq 0),$$

where $\mathfrak{E}_{\lambda,i}, \mathfrak{F}_{\lambda,i}$ are rays such that the pairs $\mathfrak{E}_{\lambda,i}, \mathfrak{F}_{\lambda,i}$ are inclined, with inclination λ , and the pairs $\mathfrak{E}_{\lambda,i}, \mathfrak{E}_{\lambda,j}; \mathfrak{E}_{\lambda,i}, \mathfrak{F}_{\lambda,j}; \mathfrak{F}_{\lambda,i}, \mathfrak{F}_{\lambda,j}$ ($i \neq j$) are orthogonal. The set of rays $\mathfrak{F}_{\lambda,i}$ may be obtained by taking the orthogonal projection in $\mathfrak{F}$ of any orthogonal set of rays $\mathfrak{E}_{\lambda,i}$ in $\mathfrak{E}_{\lambda}$; and then the sets of rays $\mathfrak{E}_{\lambda,i}$ and $\mathfrak{F}_{\lambda,i}$ are the orthogonal projections of each other in $\mathfrak{E}, \mathfrak{F}$.

If $\lambda \neq 0$ then

$$\mathfrak{E}_{\lambda} \subset \mathbf{e}\mathfrak{F}, \quad \mathfrak{F}_{\lambda} \subset \mathbf{f}\mathfrak{E}.$$

By Theorem 5.2, $\mathfrak{E}_{\lambda}$ and $\mathfrak{F}_{\lambda}$ are of dimension m_{λ} ; and by Theorem 5.4, the pairs of spaces $\mathfrak{E}_{\lambda}, \mathfrak{E}_{\mu}$ and $\mathfrak{F}_{\lambda}, \mathfrak{F}_{\mu}$ ($\lambda, \mu \neq 0, \lambda \neq \mu$) are orthogonal. By Theorem 5.2, Corollary, $\mathbf{e}\mathfrak{F}$ and $\mathbf{f}\mathfrak{E}$ are of dimension $r = \sum_{\lambda \neq 0} m_{\lambda}$. It follows now that

$$\mathbf{e}\mathfrak{F} = \sum_{\lambda \neq 0} \mathfrak{E}_{\lambda}, \quad \mathbf{f}\mathfrak{E} = \sum_{\lambda \neq 0} \mathfrak{F}_{\lambda}.$$

Further, by Theorems 3.5 and 3.6,

$$\mathfrak{E} = \mathbf{e}\mathfrak{F} \oplus \mathfrak{E}_0, \quad \mathfrak{F} = \mathbf{f}\mathfrak{E} \oplus \mathfrak{F}_0,$$

where $\mathfrak{E}_0$ is orthogonal to $\mathbf{e}\mathfrak{F}$ and $\mathfrak{F}$, and $\mathfrak{F}_0$ is orthogonal to $\mathbf{f}\mathfrak{E}$ and $\mathfrak{E}$. By Theorem 2.5, $\mathfrak{E}_1$ and $\mathfrak{F}_1$ are equal to the intersection of $\mathfrak{E}, \mathfrak{F}$ and therefore identical. Now, completing the initial scheme of decomposition, the spaces $\mathfrak{E}_{\lambda}, \mathfrak{F}_{\mu}$ ($\lambda, \mu \neq 0, \lambda \neq \mu$) are orthogonal, by Theorem 5.4.

Let $\mathbf{E}_{\lambda} = (\mathbf{E}_{\lambda,1} \dots \mathbf{E}_{\lambda,m_{\lambda}})$ be an orthonormal base for $\mathfrak{E}_{\lambda}$, and $\mathfrak{E}_{\lambda,i}$ the ray spanned by the vector $\mathbf{E}_{\lambda,i}$ ($i = 1, \dots, m$). Let $\mathbf{F}_{\lambda,i} = (\mathbf{E}'_{\lambda,i} \mathbf{f} \mathbf{E}_{\lambda,i})^{-\frac{1}{2}} \mathbf{f} \mathbf{E}_{\lambda,i}$ be a unit vector in the ray $\mathfrak{F}_{\lambda,i} = \mathbf{f}\mathfrak{E}_{\lambda,i}$ which is the orthogonal projection of $\mathfrak{E}_{\lambda,i}$ in $\mathfrak{F}$. Then, by Theorem 3.4,

$$\mathbf{e}\mathbf{f}\mathbf{E}_{\lambda} = \mathbf{E}_{\lambda}\lambda, \quad \mathbf{f}\mathbf{e}\mathbf{F}_{\lambda} = \mathbf{F}_{\lambda}\lambda,$$

and

$$\mathbf{e}\mathbf{F}_{\lambda} = \mathbf{E}_{\lambda}\lambda^{\frac{1}{2}}, \quad \mathbf{f}\mathbf{E}_{\lambda} = \mathbf{F}_{\lambda}\lambda^{\frac{1}{2}},$$

where

$$\mathbf{E}'_{\lambda}\mathbf{E}_{\lambda} = \mathbf{1}, \quad \mathbf{F}'_{\lambda}\mathbf{F}_{\lambda} = \mathbf{1}, \quad \mathbf{E}'_{\lambda}\mathbf{F}_{\lambda} = \mathbf{1}\lambda^{\frac{1}{2}},$$

and this gives the required conclusion.

Now the orthogonal projectors on $\mathfrak{E}_{\lambda}, \mathfrak{F}_{\lambda}$ are

$$\mathbf{e}_{\lambda} = \mathbf{E}_{\lambda}\mathbf{E}'_{\lambda}, \quad \mathbf{f}_{\lambda} = \mathbf{F}_{\lambda}\mathbf{F}'_{\lambda},$$

and the non-zero characteristic values of

$$\mathbf{e}_{\lambda}\mathbf{f}_{\lambda} = \mathbf{E}_{\lambda}\mathbf{E}'_{\lambda}\mathbf{F}_{\lambda}\mathbf{F}'_{\lambda}$$

are the same as those of

$$\mathbf{E}'_{\lambda}\mathbf{F}_{\lambda}\mathbf{F}'_{\lambda}\mathbf{E}_{\lambda} = \mathbf{1}\lambda,$$

which is to say they are all the same and equal to λ . Therefore the canonical angles between $\mathfrak{E}_{\lambda}, \mathfrak{F}_{\lambda}$ are equal, or these spaces are isogonal.

COROLLARY 9.11. $\mathbf{e} = \sum_{\lambda} \mathbf{e}_{\lambda}$, $\mathbf{f} = \sum_{\lambda} \mathbf{f}_{\lambda}$, where $\mathbf{e}_{\lambda}$, $\mathbf{f}_{\lambda}$ are orthogonal projectors such that

$$\mathbf{e}_0 \mathbf{f}_0 = \mathbf{0}, \quad \mathbf{e}_1 = \mathbf{f}_1; \quad \mathbf{e}_{\lambda} \mathbf{e}_{\mu} = \mathbf{0}, \quad \mathbf{e}_{\lambda} \mathbf{f}_{\mu} = \mathbf{0}, \quad \mathbf{f}_{\lambda} \mathbf{f}_{\mu} = \mathbf{0} \quad (\lambda \neq \mu);$$

$$\text{trace } \mathbf{e}_{\lambda} = \text{trace } \mathbf{f}_{\lambda} = m_{\lambda}, \quad \text{trace } \mathbf{e}_{\lambda} \mathbf{f}_{\lambda} = \lambda m_{\lambda} \quad (\lambda \neq 0).$$

COROLLARY 9.12. In any pair of spaces $\mathfrak{E}$, $\mathfrak{F}$ there exist bases $\mathbf{E}$, $\mathbf{F}$ such that

$$\mathbf{E}'\mathbf{E} = \mathbf{1}, \quad \mathbf{F}'\mathbf{F} = \mathbf{1}, \quad \mathbf{E}'\mathbf{F} = \begin{pmatrix} \rho_1 & \mathbf{0} & & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & \rho_r & \\ & \mathbf{0} & & \mathbf{0} \end{pmatrix}.$$

10. Reciprocal and extremal conditions.

THEOREM 10.1. The orthogonal projection of a given vector or ray in a given space is the unique vector or ray in the space which has extreme distance or coefficient of inclination to the given vector or ray, respectively, and the extreme is a minimum.

Let $\mathbf{Z}_0$ be the given vector, or let $\mathbf{Z}_0$ be a vector spanning the given ray. Then $\mathbf{X}_0 = \mathbf{I}_{\mathbf{E}_0} \mathbf{Z}_0$ is the orthogonal projection of the vector in the given space $\mathfrak{E}_0$, with base $\mathbf{E}_0$, or $\mathbf{X}_0$ is a vector which spans the orthogonal projection of the given ray in the given space, and for any such vector $\mathbf{X}$, $\mathbf{I}_{\mathbf{X}} \mathbf{Z}_0 = \mathbf{I}_{\mathbf{E}_0} \mathbf{Z}_0$.

If $\mathbf{X} = \mathbf{E}_0 \mathbf{x}$ is any vector in $\mathfrak{E}_0$, the square of the distance between $\mathbf{Z}_0$ and $\mathbf{X}$, and the inclination between the rays spanned by these vectors, are given by

$$(\mathbf{Z}_0 - \mathbf{X})'(\mathbf{Z}_0 - \mathbf{X}), \quad (\mathbf{Z}_0' \mathbf{X})^2 / (\mathbf{Z}_0' \mathbf{Z}_0)(\mathbf{X}' \mathbf{X}).$$

They are extreme just if

$$(\mathbf{Z}_0 - \mathbf{X})' \delta \mathbf{X} = \mathbf{0}, \quad \mathbf{Z}_0' \delta \mathbf{X} - (\mathbf{Z}_0' \mathbf{X}) \mathbf{X}' \delta \mathbf{X} / (\mathbf{X}' \mathbf{X}) = \mathbf{0},$$

for all $\delta \mathbf{x}$, where $\delta \mathbf{X} = \mathbf{E}_0 \delta \mathbf{x}$; and these conditions are equivalent to

$$\mathbf{E}_0' \mathbf{X} = \mathbf{E}_0' \mathbf{Z}_0, \quad \mathbf{E}_0' \mathbf{I}_{\mathbf{X}} \mathbf{Z}_0 = \mathbf{E}_0' \mathbf{Z}_0,$$

and then to

$$\mathbf{X} = \mathbf{I}_{\mathbf{E}_0} \mathbf{Z}_0, \quad \mathbf{I}_{\mathbf{X}} \mathbf{Z}_0 = \mathbf{I}_{\mathbf{E}_0} \mathbf{Z}_0,$$

as required. It follows that $\mathbf{X}_0$, and the ray spanned by $\mathbf{X}_0$, are uniquely determined by the stated extremal conditions with $\mathbf{Z}_0$ and the ray through $\mathbf{Z}_0$; and it is evident that the extremes are minima.

THEOREM 10.2. The rays in a given pair of spaces which have extreme inclination to each other are the reciprocal pairs of rays in those spaces.

That is, the extremely inclined rays are the rays which are orthogonal projections of each other in the spaces; and these have been seen to be the pairs of rays spanned by the reciprocal pairs of latent vectors of the products of the orthogonal projectors on the spaces. The proof is that if the inclination is extreme for variations of the rays in both the spaces, then it is extreme for variations in just one space or the other, so it follows, by Theorem 9.1, that the rays are orthogonal projections of each other in those spaces.

The direct consideration of the extremal condition gives an alternative procedure of computation. Thus, let $\mathbf{X} = \mathbf{E} \mathbf{x}$, $\mathbf{Y} = \mathbf{F} \mathbf{y}$ be unit vectors spanning extremely inclined rays. Then

$$\delta(\mathbf{x}' \mathbf{E}' \mathbf{F} \mathbf{y}) = 0$$

for all variations $\delta \mathbf{x}$, $\delta \mathbf{y}$ of $\mathbf{x}$, $\mathbf{y}$ under the constraints

$$\mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = 1, \quad \mathbf{y}'\mathbf{F}'\mathbf{F}\mathbf{y} = 1.$$

Equivalently,

$$\mathbf{E}'\mathbf{F}\mathbf{y} = \lambda \mathbf{E}'\mathbf{E}\mathbf{x}, \quad \mathbf{F}'\mathbf{E}\mathbf{x} = \mu \mathbf{F}'\mathbf{F}\mathbf{y},$$

where λ , μ are the Lagrangian multipliers corresponding to the constraints. Then

$$\lambda = \mathbf{x}'\mathbf{E}'\mathbf{F}\mathbf{y} = \mu.$$

But, with $\lambda = \mu$, one of the constraints becomes redundant, so there is the following conclusion.

THEOREM 10.3. *A necessary and sufficient condition for $\rho = \mathbf{x}'\mathbf{E}'\mathbf{F}\mathbf{y}$ to be extreme for variations of $\mathbf{x}$, $\mathbf{y}$ under the constraints $\mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = 1$, $\mathbf{y}'\mathbf{F}'\mathbf{F}\mathbf{y} = 1$ is*

$$\begin{pmatrix} -\rho \mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho \mathbf{F}'\mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{0}, \quad \mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = 1.$$

COROLLARY 10.31. *The extreme values ρ are the roots of the equation*

$$\begin{vmatrix} -\rho \mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho \mathbf{F}'\mathbf{F} \end{vmatrix} = 0.$$

It has been seen directly, in Theorem 3.4, that these roots squared are the roots of the equation $|\rho^2 \mathbf{1} - \mathbf{I}_E \mathbf{I}_F| = 0$, with the same multiplicities.

THEOREM 10.4. *The nullity of the matrix $\mathbf{K}_\rho = \begin{pmatrix} -\rho \mathbf{E}'\mathbf{E} & \mathbf{E}'\mathbf{F} \\ \mathbf{F}'\mathbf{E} & -\rho \mathbf{F}'\mathbf{F} \end{pmatrix}$ is the multiplicity of ρ as a root of the equation $|\mathbf{K}_\rho| = 0$.*

This can be proved indirectly from results already established, or directly as follows. If $\mathbf{U} = \mathbf{E}\boldsymbol{\alpha}$, $\mathbf{V} = \mathbf{F}\boldsymbol{\beta}$ are equivalent orthonormal bases, let

$$\mathbf{W}_\rho = \begin{pmatrix} -\rho \mathbf{1} & \mathbf{U}'\mathbf{V} \\ \mathbf{V}'\mathbf{U} & -\rho \mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta} \end{pmatrix}.$$

Then $\boldsymbol{\gamma}'\mathbf{K}_\rho\boldsymbol{\gamma} = \mathbf{W}_\rho$, so that $\mathbf{K}_\rho$, $\mathbf{W}_\rho$ have the same nullity, say n_ρ , and $|\mathbf{K}_\rho| = 0$, $|\mathbf{W}_\rho| = 0$ have the same roots, thus ρ with multiplicity m_ρ . But m_ρ is now the multiplicity of ρ as a characteristic value of the symmetric matrix $\mathbf{W}_\rho$, by which it follows that $m_\rho = n_\rho$.

If $\begin{pmatrix} \mathbf{x}_\rho \\ \mathbf{y}_\rho \end{pmatrix}$ denotes any solution corresponding to a root ρ , it follows that there exist m_ρ such independent solutions, and these can be chosen so as to obtain m_ρ orthogonal vectors $\begin{pmatrix} \mathbf{X}_{\rho,i} \\ \mathbf{Y}_{\rho,i} \end{pmatrix}$ ($i = 1, \dots, m_\rho$), where $\mathbf{X}_\rho = \mathbf{E}\mathbf{x}_\rho$, $\mathbf{Y}_\rho = \mathbf{F}\mathbf{y}_\rho$. Now for any such vectors, belonging to roots ρ , σ , evidently

$$\mathbf{Y}'_\sigma \mathbf{Y}_\rho = \rho \mathbf{X}'_\sigma \mathbf{X}_\rho, \quad \mathbf{Y}'_\sigma \mathbf{X}_\rho = \rho \mathbf{X}'_\sigma \mathbf{Y}_\rho,$$

so that

$$\mathbf{X}'_\sigma \mathbf{X}_\rho = \mathbf{X}'_\sigma \mathbf{Y}_\rho = \sigma \mathbf{Y}'_\sigma \mathbf{Y}_\rho.$$

It follows that if $\mathbf{X}'_\sigma \mathbf{X}_\rho = 0$, then, provided ρ , $\sigma \neq 0$, $\mathbf{X}'_\rho \mathbf{X}_\sigma = 0$, $\mathbf{X}'_\rho \mathbf{Y}_\sigma = 0$ and $\mathbf{Y}'_\sigma \mathbf{Y}_\rho = 0$. Also it appears that $\mathbf{X}'_\sigma \mathbf{X}_\rho = 0$ automatically if $\rho \neq \sigma$; and $\mathbf{X}'_\sigma \mathbf{X}_\rho = \mathbf{Y}'_\sigma \mathbf{Y}_\rho$ if $\sigma = \rho$. It is now seen that there are found m_ρ orthogonal unit vectors $\mathbf{X}_{\rho,i}$ in $\mathfrak{E}$ and m_ρ orthogonal unit vectors $\mathbf{Y}_{\rho,i}$ in $\mathfrak{F}$ ($i = 1, \dots, m_\rho$), for each non-zero root ρ , such that

$$\mathbf{X}'_{\sigma,i} \mathbf{Y}_{\rho,j} = 0 \quad \text{if } \sigma \neq \rho \quad \text{or } i \neq j \quad \text{and} \quad \mathbf{X}'_{\sigma,i} \mathbf{Y}_{\sigma,i} = \rho.$$

The final theorem, which is easily verified, shows explicitly the equivalence of the three conditions formulated between pairs of vectors in the spaces, that they have extreme inclination, that they span rays which are orthogonal projections of each other, and that they are latent vectors of the products of the orthogonal projectors; and it shows how the two processes of calculation, in terms of latent vectors, and in terms of solutions of the extremality equations, are transformed into each other.

THEOREM 10.5. *Provided $\rho \neq 0$, the equations*

$$\mathbf{E}'\mathbf{F}\mathbf{y} = \rho\mathbf{E}'\mathbf{E}\mathbf{x}, \quad \mathbf{F}'\mathbf{E}\mathbf{x} = \rho\mathbf{F}'\mathbf{F}\mathbf{y}, \quad \mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = 1$$

are equivalent to the equations

$$\mathbf{e}\mathbf{Y} = \mathbf{X}\rho, \quad \mathbf{f}\mathbf{X} = \mathbf{Y}\rho, \quad \mathbf{X}'\mathbf{X} = 1,$$

and to the equations

$$\mathbf{e}\mathbf{f}\mathbf{X} = \mathbf{X}\rho^2, \quad \mathbf{Y} = (\mathbf{X}'\mathbf{f}\mathbf{X})^{-1}\mathbf{f}\mathbf{X}, \quad \mathbf{X}'\mathbf{X} = 1,$$

and to the other sets of equations obtained by interchanging $\mathbf{X} = \mathbf{E}\mathbf{x}$ and $\mathbf{Y} = \mathbf{F}\mathbf{y}$ in these, and they imply $\mathbf{X}'\mathbf{Y} = \rho$.

For general accounts of the theory of projectors, reference is made to Hamburger and Grimshaw (1) and to Halmos (2). Material in Afriat (3), with certain improvements, has here been included for the sake of completeness.

This paper is a development of mathematical features of some statistical work done at the Department of Applied Economics, Cambridge. Its origin is in an attempt to make concise the algebra which appeared in an investigation of methods for the seasonal correction of economic time series; and I wish to express with thanks my indebtedness to Prof. J. R. N. Stone, both for introduction to his work in this subject, and also for allowing the opportunity for the development of the present material. A statement of method for the analysis of seasonal variation, from the general point of view of multivariate analysis towards which this paper is directed, is in course of preparation. Also I have to acknowledge my indebtedness to my colleagues, especially Mr Arthur Adams, Mr Alan Brown and Mr John Aitchison, for discussions which did much to elicit these ideas; and to Dr R. Penrose, with whom I had discussion of the algebraical material, and who made valuable suggestions, especially in regard to the lemmas of § 6, which are results of our collaboration.

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