

A SUPERPOSITION THEOREM FOR BOUNDED CONTINUOUS FUNCTIONS

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ABSTRACT. It is shown that there exist $2n + 1$ real valued, continuous functions $\phi_0, \dots, \phi_{2n}$ defined on $\mathbf{R}^n$ such that every bounded real valued continuous function on $\mathbf{R}^n$ is expressible in the form $\sum_{i=0}^{2n} g \circ \phi_i$ for some $g \in C(\mathbf{R})$. Extensions to some unbounded functions are also made.

The Kolmogorov superposition theorem states, in the form of Lorentz [4], that there exist $2n + 1$ real valued functions $\phi_1, \dots, \phi_{2n+1}$ continuous on the closed unit cube, I^n , in $\mathbf{R}^n$ such that for any $f \in C(I^n)$ there is $g \in C(\mathbf{R})$ such that

$$(*) \quad f(x) = \sum_{i=1}^{2n+1} g(\phi_i(x)) \quad \text{for all } x \in I^n.$$

In fact one may take ϕ_i to be of the form $\phi_i(x_1, \dots, x_n) = \sum_{p=1}^n e^p \psi_i(x_p)$ where $\psi_i \in C(I)$. The various proofs of this result, e.g., [2], [3], [4], all make use of the uniform continuity of f . Recently, Doss [1], has proven the following theorem.

THEOREM. *There are functions $\psi_1, \dots, \psi_{2n-1}, \phi_1, \dots, \phi_{2n+1} \in C(\mathbf{R}^n)$ such that for any $f \in C(\mathbf{R}^n)$ there exist $g, h \in C(\mathbf{R})$ such that*

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{i=1}^{2n+1} g(\phi_i(x)) \quad \text{for all } x \in \mathbf{R}^n.$$

Our main result is that there is a choice of $\phi_1, \dots, \phi_{2n+1}$ such that for bounded functions (and some unbounded ones) we may take $h \equiv 0$. It has as an immediate corollary that every $f \in C(\mathbf{R}^n)$ can be represented in terms of $\phi_1, \dots, \phi_{2n+1}$, the tangent function, and a function of a single variable. In the spirit of Hedberg [2], we obtain the existence of the ϕ_i 's by means of a category argument. We then show that if f has compact support, then the representing function g must decay exponentially to zero away from the support of f . Paracompactness of $\mathbf{R}^n$ enables us to complete the proof. The norm used on functions is always the uniform norm on $\mathbf{R}^n$ unless stated otherwise.

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Throughout this paper $n \geq 2$ is a fixed but arbitrary integer and for $m \geq 0$ we define the following subsets of $\mathbf{R}^n$: $B_m = \{x: \|x\| \leq m\}$, $S_m = \{x: \|x\| = m\}$, and $R_m = \{x: m-1 \leq \|x\| \leq m\}$; here we use the l_∞ norm. We now recall the cubes of Kolmogorov, [3], [4].

LEMMA 1. *There exist closed, bounded cubes $\{c_{i,k}^q: 1 \leq q \leq 2n+1, k \geq 1, i \in \mathbf{N}\}$ and strictly decreasing null sequences $\{a_k\}_{k \geq 1}, \{b_k\}_{k \geq 1}$ such that*

(1) *For fixed k each point of $\mathbf{R}^n$ is contained in at least $n+1$ of the cubes $\{c_{i,k}^q: 1 \leq q \leq 2n+1, i \in \mathbf{N}\}$.*

(2) *For fixed k and q , the cubes $\{c_{i,k}^q: i \in \mathbf{N}\}$ are disjoint.*

(3) *$\text{diam}(c_{i,k}^q) \leq a_k$ for all i, q .*

(4) *$\min_i H(c_{j,k}^q, c_{i,k}^q) \leq b_k$ for all j, q where $H(A, B)$ denotes the Hausdorff distance between A and B .*

The next result shows that the families of cubes can be separated by functions with prescribed ranges.

LEMMA 2. *There are $2n+1$ functions $\phi_1, \dots, \phi_{2n+1}$ in $C(\mathbf{R}^n)$ such that*

(a) *for $1 \leq q \leq 2n+1$ and $m \geq 1$, $\phi_q(R_m) = [m, m+2]$ and $\phi_q(S_m) = [m+1, m+2]$.*

(b) *for each $N \geq 1$, there is a $k \geq N$ such that the sets $\phi_q(c_{i,k}^q)$, $1 \leq q \leq 2n+1, i \in \mathbf{N}$, are mutually disjoint.*

PROOF. The set $M = \{(\psi_1, \dots, \psi_{2n+1}) \in [C(\mathbf{R}^n)]^{2n+1}: \text{each } \psi_q \text{ satisfies (a)}\}$ is a nonempty complete metric space if it is given the product-compact-open topology. For $k \geq 1$ let $\Theta_k = \{(\psi_q) \in M: \text{the sets } \psi_q(c_{i,k}^q), 1 \leq q \leq 2n+1, i \in \mathbf{N} \text{ are mutually disjoint}\}$. It can be verified that for each N , $\bigcup_{k \geq N} \Theta_k$ is open and dense in M . By the Baire category theorem $\bigcap_{N \geq 1} \bigcup_{k \geq N} \Theta_k$ is nonvoid; this is statement (b). Q.E.D.

LEMMA 3. *Let $f \in C(\mathbf{R}^n)$ with $\text{supp } f \subseteq \text{int}(\bigcup_{j=0}^l R_{m+j})$ and let $n(n+1)^{-1} < \theta < 1$. There exists $g \in C(\mathbf{R})$ such that*

(a) *$\|g\| \leq (n+1)^{-1} \|f\|$,*

(b) *$|f(x) - \sum_q g(\phi_q(x))| < \theta \|f\|$, for all $x \in \mathbf{R}^n$ and*

(c) *$\text{supp } g \subseteq [m-1, m+l+3]$.*

PROOF. Following Lorentz [4, p. 173], we let $\varepsilon > 0$ be such that $n(n+1)^{-1} + \varepsilon \leq \theta$. Let N be so large that if $\|x - y\| < a_N$ (cf. Lemma 1), then $|f(x) - f(y)| < \varepsilon \|f\|$. Let $k \geq N$ be such that the sets $\phi_q(c_{i,k}^q)$, $1 \leq q \leq 2n+1, i \in \mathbf{N}$, are mutually disjoint. We define g to be constant on each set $\phi_q(c_{i,k}^q)$ the constant being the value of $(n+1)^{-1}f$ at the center of $c_{i,k}^q$. Extending g linearly to all of $\mathbf{R}$ we see that (a) is satisfied. The proof of (b) follows as in [4]. To prove (c) note that $g(\phi_q(c_{i,k}^q)) \neq \{0\}$ only if $c_{i,k}^q \cap (\bigcup_{j=0}^l R_{m+j}) \neq \emptyset$ and use part (a) of Lemma 2.

The next result shows that if $f(x)$ has compact support, then it can be represented in the form (*) where the function $g(x)$ decays to zero exponentially as $x \rightarrow \infty$.

LEMMA 4. Let $f \in C(\mathbf{R}^n)$ with $\text{supp } f \subseteq \text{int}(R_m \cup R_{m+1})$ and let θ be as in Lemma 3. There exists $g \in C(\mathbf{R})$ and a constant C depending on only θ such that

- (a) for all $x \in \mathbf{R}^n$, $f(x) = \sum_q g(\phi_q(x))$ and
 (b) $\|g\|_{L_\infty[k, k+1]} \leq C \|f\| \theta^{|m-k|}$ for all $k \geq 0$.

PROOF. Let g_1 satisfy the conditions of Lemma 3 and define $f_1(x) = f(x) - \sum_i g_1(\phi_i(x))$. Note that $\text{supp } f_1 \subseteq \bigcup_{j=-3}^4 R_{m+j}$. In general, if f_k is defined, we let g_{k+1} be such that $\|g_{k+1}\| \leq (n+1)^{-1} \|f_k\|$ and $|f_k(x) - \sum_i g_{k+1}(\phi_i(x))| < \theta \|f_k\|$ and define $f_{k+1}(x) = f_k(x) - \sum_i g_{k+1}(\phi_i(x))$. As in [4], we have $\|f_k\| \leq \theta^k \|f\|$ and $\|g_{k+1}\| \leq \theta^k \|f\|$ which yield (a). By Lemma 3 the g_k 's can be chosen so that $\text{supp } f_k \subseteq \bigcup_{j=-3k}^{3k+1} R_{m+j}$ and $\text{supp } g_{k+1} \subseteq [m-3k-1, m+3k+4] \cap [0, \infty)$. Consequently, on any interval of the form $[0, m-3k-1]$ or $[m+3k+4, \infty)$ the norm of g is bounded by $\|f\| \sum_{r=k+1}^\infty \theta^r$. This implies (b). Q.E.D.

THEOREM 1. Let $f \in C(\mathbf{R}^n)$ be bounded. There exists $g \in C(\mathbf{R})$ such that for all $x \in \mathbf{R}^n$, $f(x) = \sum_{i=1}^{2n+1} g(\phi_i(x))$.

PROOF. Since $\mathbf{R}^n$ is paracompact, we may write $f(x) = \sum_{m=0}^\infty f_m(x)$ where $\text{supp } f_m \subseteq \text{int}(R_m \cup R_{m+1})$ and $\|f_m\| \leq \|f\|_{L_\infty(B_{m+1})}$. Let g_m satisfy the conclusions of Lemma 4 for f_m . So,

$$f(x) = \sum_{m=0}^\infty \sum_{i=1}^{2n+1} g_m(\phi_i(x)), \quad x \in \mathbf{R}^n,$$

and

$$\|g_m\|_{L_\infty[k, k+1]} \leq C \theta^{|m-k|} \|f\|_{L_\infty(B_{m+1})}.$$

By the Weierstrass M -test, $\sum_{m=0}^\infty g_m \in C[k, k+2]$ for all $k \geq 0$. Therefore, $\sum_{m=0}^\infty g_m \in C(\mathbf{R})$ and $f(x) = \sum_{i=1}^{2n+1} (\sum_{m=0}^\infty g_m(\phi_i(x)))$. Q.E.D.

COROLLARY 1. Let $f \in C(\mathbf{R}^n)$ and assume that $\sum_{m=0}^\infty \theta^m \|f\|_{L_\infty(B_m)}$ converges for some $n(n+1)^{-1} < \theta < 1$, then there is $g \in C(\mathbf{R})$ such that

$$f = \sum_{i=1}^{2n+1} g \circ \phi_i.$$

PROOF. The assumptions insure that the series $\sum_{m=0}^\infty g_m$ converges uniformly on compact subsets of $\mathbf{R}$.

We have not been able to extend Theorem 1 to all unbounded functions. However, it is possible to represent unbounded functions by nonlinear superpositions of the ϕ_i 's.

COROLLARY 2. There is a constant $K > 0$ such that for $f \in C(\mathbf{R}^n)$, there exists $g \in C(\mathbf{R})$ with $\|g\| \leq K$ and $f(x) = \tan\{\sum_{i=1}^{2n+1} g(\phi_i(x))\}$ for all $x \in \mathbf{R}^n$.

PROOF. Since $f(x) = \tan\{\text{Arc tan } f(x)\}$, apply Theorem 1 to $\text{Arc tan } f(x)$.

REMARK. Theorem 1 can be extended to more general spaces than $\mathbf{R}^n$. For example, let Ω be an open connected subset of $\mathbf{R}^n$. In the arguments used

above, replace R_m by $\{x \in \Omega: 1/m \leq \text{dist}(x, \partial\Omega) \leq 1/(m-1)\}$, S_m by $\{x \in \Omega: \text{dist}(x, \partial\Omega) = 1/m\}$ and B_m by $\{x \in \Omega: 1/m \leq \text{dist}(x, \partial\Omega)\}$.

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